Effect of fast velocity on the beam dynamics in a modified betatron accelerator

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In this paper we have derived the equations that describe the slow motion (bounce) of the reference electron that is located at the beam centroid, taking into account its fast motion, for an electron beam that is confined in the magnetic field of a Modified Betatron Accelerator. The two equations for the guiding center (g.c.) of the centroid have been derived by averaging the centroid orbit over two time periods that are distinctly different. These two periods are associated with the normal modes of the system. Although our approach is not exact, it has the advantage that the fast velocity appears explicitly in the various expressions and thus its significance on the centroid dynamics can be rather easily assessed. A general conclusion of our results is that the fast velocity does not have a profound impact on the dynamics of the centroid, provided that the ratio $u_\perp/u_0 \lesssim 0.5$, where $u_\perp$ is the transverse and u_0 is the total velocity. A very interesting conclusion of our work is that the centroid equilibrium position is insensitive to $u_\perp$, provided that $u_\perp \lesssim u_0/2$. We explain this counter-intuitive result. [S1063-651X(97)13305-0]

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I. INTRODUCTION

This paper reports on some implications of fast motion on the dynamics of the beam centroid, when the electron beam is confined in a Modified Betatron Accelerator (MBA). The MBA [1,2] is a high current, recirculating, induction accelerator with high effective accelerating gradient. Its improved current carrying capability is due to the addition of a toroidal B_θ field and a strong focusing twisted quadrapole [3,4] to the vertical B_z field of the conventional betatron [5].

The MBA was developed at the Naval Research Laboratory (NRL) [2] and the University of California [6] at Irvine. During the last phase of its operation, the trapped electron current in the NRL device [2] routinely exceeded 1 kA and the peak energy exceeded 20 MeV. All the electron rings in the NRL device were formed by an injected electron beam of energy, typically, between 0.5 and 0.6 MeV. In the Irvine device the majority of the electron rings were formed by runaway electrons and not by an injected beam.

There are numerous theoretical studies of the beam dynamics in a MBA magnetic field configuration with [4,7] and without [8] strong focusing field. In the majority of these studies the fast motion of the electrons, i.e., their motion associated with the B_θ field has been neglected. In this paper we have derived two equations that describe the slow motion (bounce) of the reference electron that is located at the beam centroid taking into account its fast motion.

To lowest order, the fast motion is associated with the gyration of the reference electron around its guiding center as a result of the toroidal magnetic field. In contrast, the bounce motion is the slow gyration of the guiding center and is due to a combination of $\vec{E}\times\vec{B}$ and $\vec{B}\times\nabla\vec{B}$ drifts, where $\vec{E}$ is the induced field at the beam centroid from the image fields of the surrounding walls and $\vec{B}$ is the total magnetic field.

The two equations for the guiding center (g.c.) of the centroid have been derived by averaging the centroid orbit over two time periods that are distinctly different. These two periods are associated with the normal modes of the system. Although our approach is not exact, it has the advantage that the fast velocity appears explicitly in the various expressions and thus its significance on the centroid dynamics can be rather easily assessed.

It has been shown [9] that in a MBA with strong focusing there are four characteristic transverse modes. For modest current in the windings and modest space charge, as in the NRL device, the four modes become very simple and in the laboratory frame are given by [1] $\omega_{++}\approx m\Omega_z/\gamma+\Omega_\theta/\gamma$ (high-frequency cyclotron), $\omega_{--}\approx\omega_B$ (bounce), $\omega_{-+}\approx-\Omega_\theta/\gamma$ (cyclotron), and $\omega_{+-}\approx m\Omega_z/\gamma-\omega_B$ (SF mode), where Ω_z and Ω_θ are the B_z and B_θ cyclotron frequencies, γ is the relativistic factor, ω_B is the bounce frequency, and m is the number of field periods.

The parameters of the NRL device are such that $\omega_{-+}\gg\omega_{+-}\gg\omega_{--}$ during the initial time period that follows beam injection. This separation of frequencies is very convenient because it allows averaging of the orbit, initially over the short period $2\pi/\omega_{-+}$ and subsequently over the longer period $2\pi/\omega_{+-}$.

The orbit averaging over the short period for particles with relativistic energies has been carried out previously. When both electric field components, i.e., the transverse $\vec{E_\perp}$ and the parallel $\vec{E_\parallel}$ along the total magnetic field and $\partial/\partial t$ are small and more precisely of order ϵ, where ϵ is the expansion parameter, the guiding center drift equations are given by Northrop [10]. These equations are the starting point of our calculation.

A general conclusion of our results is that the fast velocity does not have a profound impact on the dynamics of the centroid, provided that the ratio $u_\perp/u_0\lesssim0.5$, where $u_\perp$ is the transverse and u_0 is the total velocity.

A very interesting conclusion of our work is that the centroid equilibrium position is insensitive to $u_\perp$, provided that $u_\perp \lesssim u_0/2$. Obviously, such a conclusion is counterintuitive, because as $u_\perp$ grows at the expense of u_θ, it would be expected that the guiding center would start moving inwards towards the major axis, since the force from the betatron field varies linearly with u_θ, while the centrifugal force has a quadratic dependence on u_θ. However, this is not the case because a new force is developed from the fast motion that compensates for this faster reduction of the centrifugal force (see Sec. III).

II. FAST MOTION

As stated in the Introduction, when $E_\perp$, $E_\parallel$, and $\partial/\partial t$ are of $O(\epsilon)$, the relativistic guiding center drift equations for the fast motion $(\omega_{-+}=\Omega_\theta)$ are given by Northrop [10] [Eqs. (1.68), (1.69), and (1.70)]. Since for the beam centroid $\vec{\nabla}\times\vec{B}=\vec{0}$, these equations in MKS units take the simpler form:

$$\dot{\vec{R}}_\perp=\frac{\vec{E}\times\vec{B}}{B^2}+\frac{\vec{B}\times\vec{\nabla}\vec{B}}{B^2}\left[\frac{M_r}{\gamma e}+\frac{P_\parallel^2}{m_0\gamma eB}\right], \tag{1}$$

$$\frac{1}{e}\frac{dP_\parallel}{dt}=E_\parallel-\frac{M_r}{\gamma e}\frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{B}, \tag{2}$$

and

$$M_r=P_\perp^2/2m_0B=\text{const}, \tag{3}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field at the guiding center of the fast motion, $P_\perp=\gamma m_0 u_\perp$ and $P_\parallel=\gamma m_0 u_\parallel$ are the components of momentum transverse and along the magnetic field, γ is the relativistic factor, $u_\perp$ and $u_\parallel$ are the components of the fast velocity, e is the charge of the gyrating particle, and m_0 its rest mass.

A. Applied magnetic fields

The modified betatron accelerator comprises three distinct magnetic fields: the betatron field components B_z and B_r, the toroidal B_θ, and the strong focusing field. In the system of coordinates shown in Fig. 1, the two betatron field components are given by

$$B_z=B_{z0}\left(\frac{r_0}{r}\right)^n, \tag{4}$$

$$B_r=-nB_{z0}\left(\frac{r_0}{r}\right)^n\frac{z}{r}, \tag{5}$$

where B_{z0} is the field on the minor axis, r_0 is the major radius of the device, and n is the external field index.

Similarly, the toroidal magnetic field is

$$B_\theta=B_{\theta 0}\frac{r_0}{r}, \tag{6}$$

where $B_{\theta 0}$ is the field on the minor axis.

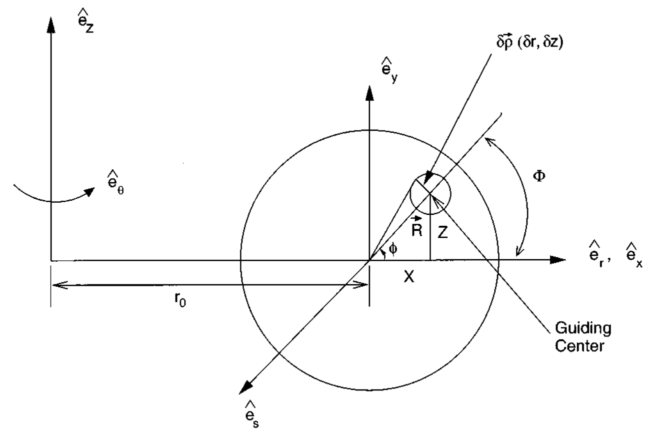

FIG. 1. Systems of coordinates.

To the lowest order and omitting toroidal corrections, the two components of the strong focusing field produced by the stellarator windings are given by [4,7]

$$B_\rho^{\rm st}=2B_s\epsilon_{\rm st}I_2(2\alpha\rho)\sin[2(\phi-\alpha s)], \tag{7}$$

$$B_\phi^{\rm st}=2\frac{B_s}{\alpha\rho}\epsilon_{\rm st}I_2(2\alpha\rho)\cos[2(\phi-\alpha s)], \tag{8}$$

where $B_s\epsilon_{\rm st}=4\alpha B_0\rho_0K_2'(2\alpha\rho_0), \alpha=2\pi/L$, L is the period of the stellarator windings, ρ_0 is the minor radius of the windings, K_2' is the derivative of the second order modified Bessel function, $B_0=2\mu_0I/L$, I is the current of the windings, and $I_2(2\alpha\rho)$ is the second order modified Bessel function.

When the beam centroid is near the minor axis, i.e., when $x/r_0 \ll 1$, $z/r_0 \ll 1$ and $4\pi\rho/L \ll 1$, the linear MBA fields are

$$B_z=B_{z0}\left(1-n\frac{x}{r_0}\right), \tag{9}$$

$$B_r=-B_{z0}n\frac{z}{r_0}, \tag{10}$$

$$B_\theta=B_{\theta 0}\left(1-\frac{x}{r_0}\right), \tag{11}$$

$$B_\rho^{\rm st}=B_s\epsilon_{\rm st}\alpha\rho\sin[2(\phi-\alpha s)], \tag{12}$$

and

$$B_\phi^{\rm st}=B_s\epsilon_{\rm st}\alpha\rho\cos[2(\phi-\alpha s)]. \tag{13}$$

Since

$$B_r^{\rm st}=B_\rho^{\rm st}\cos\phi-B_\phi^{\rm st}\sin\phi,$$

$$B_z^{\rm st}=B_\rho^{\rm st}\sin\phi+B_\phi^{\rm st}\cos\phi,$$

and $s=-r_0\theta$, Eqs. (12) and (13) become

$$B_r^{\rm st}=B_s\epsilon_{\rm st}\alpha[z\cos(2\alpha r_0\theta)+x\sin(2\alpha r_0\theta)], \tag{14}$$

$$B_z^{\rm st}=B_s\epsilon_{\rm st}\alpha[x\cos(2\alpha r_0\theta)-z\sin(2\alpha r_0\theta)]. \tag{15}$$

For particles with relativistic energies, the velocity u_θ is approximately constant. This allows ω_w to be written as a function of toroidal variable θ, i.e., $\omega_w = 2\alpha u_\theta = -2\alpha s/t = 2\alpha r_0 \theta/t$.

Substituting the last relation into Eqs. (14) and (15), we obtain

$$B_r^{\mathrm{st}} = B_s \epsilon_{\mathrm{st}} \alpha (z\cos\omega_w t + x\sin\omega_w t), \tag{16}$$

$$B_z^{\mathrm{st}} = B_s \epsilon_{\mathrm{st}} \alpha (x\cos\omega_w t - z\sin\omega_w t). \tag{17}$$

Equations (9), (10), (11), (16), and (17) give the linear external magnetic fields of the MBA. These fields together with the image fields produced by the induced charge and current on the walls of the torus provide the forces acting on the beam centroid and determine the drift motion of its guiding center through Eqs. (1) and (2).

B. Image fields at the ring centroid

An accurate self-consistent determination of self-fields of a high-current electron ring in a modified betatron configuration that is supplemented by strong focusing is very difficult, because the minor cross section of the ring varies along the toroidal direction.

However, here we are interested in the macroscopic motion of the ring and therefore in the image fields that act on the ring centroid. These fields are not sensitive to the detailed shape of the ring cross section and thus can be computed approximately by assuming that the ring has a circular cross section that is uniformly filled. Neglecting toroidal corrections, the fields at the reference particle are [7]

$$B_\phi = \frac{\mu_0 e n_0 u_\theta \pi r_b^2 \rho}{2\pi a^2 (1-\rho^2/a^2)}, \tag{18}$$

$$E_\rho = -\frac{e n_0 \pi r_b^2 \rho}{2\pi \epsilon_0 a^2 (1-\rho^2/a^2)}, \tag{19}$$

where r_b is the minor radius of the ring, a is the minor radius of a perfectly conducting torus, and n_0 is the uniform electron density.

Since

$$B_r = -B_\phi \sin\phi, \quad E_r = E_\rho \cos\phi, \tag{20}$$

and

$$B_z = B_\phi \cos\phi, \quad E_z = E_\rho \sin\phi, \tag{21}$$

the linear $(\rho/a \ll 1)$ effective electric field components are

$$E_r^{\mathrm{eff}} = E_r + u_\theta B_z = -\frac{e N_l}{2\pi\epsilon_0 a \gamma_\parallel^2}\frac{x}{a}, \tag{22}$$

and

$$E_z^{\mathrm{eff}} = E_z - u_\theta B_r = -\frac{e N_l}{2\pi\epsilon_0 a \gamma_\parallel^2}\frac{z}{a}, \tag{23}$$

where N_l is the number of electrons per unit length and $\gamma_\parallel = 1/\sqrt{1-(u_\parallel/c)^2}$.

C. Drift velocities

The guiding center drift velocity components can be computed from

$$\vec{u}_r = (\dot{\vec{R}}_\perp + \vec{u}_\parallel)\cdot\hat{e}_r, \quad \vec{u}_z = (\dot{\vec{R}}_\perp + \vec{u}_\parallel)\cdot\hat{e}_z,$$

$$\vec{u}_\theta = (\dot{\vec{R}}_\perp + \vec{u}_\parallel)\cdot\hat{e}_\theta, \tag{24}$$

where $\vec{u}_\parallel = u_\parallel \hat{e}_\parallel$, $\hat{e}_\parallel = (B_r\hat{e}_r + B_z\hat{e}_z + B_\theta\hat{e}_\theta)/B$, B_r is the total radial, B_z is the total vertical, B_θ is the total toroidal external field components, and B is the total magnetic field.

Substituting Eq. (1) into Eqs. (24), we obtain

$$u_r = \frac{(\vec{E}\times\vec{B})_r}{B^2} + \frac{(\vec{B}\times\vec{\nabla}\vec{B})_r}{B^2}\left[\frac{M_r}{\gamma e} + \frac{P_\parallel^2}{m_0\gamma e B}\right] + \frac{P_\parallel}{m_0\gamma}\frac{B_r}{B}, \tag{25}$$

$$u_z = \frac{(\vec{E}\times\vec{B})_z}{B^2} + \frac{(\vec{B}\times\nabla\vec{B})_z}{B^2}\left[\frac{M_r}{\gamma e} + \frac{P_\parallel^2}{m_0\gamma e B}\right] + \frac{P_\parallel}{m_0\gamma}\frac{B_z}{B}, \tag{26}$$

$$u_\theta = \frac{(\vec{E}\times\vec{B})_\theta}{B^2} + \frac{(\vec{B}\times\nabla\vec{B})_\theta}{B^2}\left[\frac{M_r}{\gamma e} + \frac{P_\parallel^2}{m_0\gamma e B}\right] + \frac{P_\parallel}{m_0\gamma}\frac{B_\theta}{B}. \tag{27}$$

During the first phase of acceleration, i.e., during the time period that follows injection of the beam, B_θ is the dominant field and therefore $B \approx B_\theta$. Omitting second order terms, such as $(B_z/B)^2$, $(B_s/B)^2 \cdots$ or higher, Eqs. (25), (26), and (27) become

$$\frac{r_0\Omega_{\theta 0}}{\gamma}u_r = \left[\frac{2\nu c^2}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{a}\right)^2 - \frac{r_0\Omega_{z0} n u_\parallel}{\gamma}\right]\left(\frac{z}{r_0}\right) + \frac{u_\parallel \alpha r_0 \Omega_s \epsilon_{\mathrm{st}}}{\gamma} \times (z\cos\omega_w t + x\sin\omega_w t), \tag{28}$$

$$\frac{r_0\Omega_{\theta 0}}{\gamma}u_z = \left[u_\parallel^2 - \frac{u_\parallel\Omega_{z0} r_0}{\gamma}n - \frac{2\nu c^2}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{a}\right)^2 + \frac{\Omega_{\theta 0} M_r}{\gamma^2 e}\right] \times \left(\frac{x}{r_0}\right) + u_\parallel\left(r_0\frac{\Omega_{z0}}{\gamma} - u_\parallel\right) - \frac{\Omega_{\theta 0} M_r}{\gamma^2 e} + \frac{u_\parallel \alpha r_0 \Omega_s \epsilon_{st}}{\gamma}(x\cos\omega_w t - z\sin\omega_w t), \tag{29}$$

and

$$u_\parallel \approx u_\theta, \tag{30}$$

where ν is the Budker parameter.

Equations (28), (29), and (30) give the drift velocity components and the longitudinal velocity component of the guiding center of the beam centroid that is associated with the fast motion. To obtain the bounce motion of the guiding center that is associated with the intermediate motion, Eqs. (28), (29), and (30) will be time averaged over a period $2\pi/\omega_w$.

III. INTERMEDIATE MOTION

In the system of coordinates shown in Fig. 1, the instantaneous position of the guiding center of the fast motion is

$$r = r_0 + X + \delta r, \tag{31}$$

$$z = Z + \delta z, \tag{32}$$

where r_0 is the major radius of the torus, X and Z indicate the position of the guiding center of the intermediate motion, relative to the minor axis, and δr and δz indicate the position of the guiding center of the fast motion relative to that of the intermediate motion.

Introducing the complex variables

$$u = r + iz, \tag{33}$$

$$U = X + iZ, \tag{34}$$

$$\delta u = \delta r + i\,\delta z, \tag{35}$$

and combining Eqs. (31)–(35), we obtain

$$u = r_0 + U + \delta u. \tag{36}$$

Multiplying Eq. (29) by i and adding it to Eq. (28) and following the same procedure as in Ref. [7], we obtain

$$\frac{\Omega_{\theta 0} r_0}{c^2\gamma}\dot{X} = \left[\frac{2\nu}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{\alpha}\right)^2 - \frac{u_\parallel \Omega_{z0} r_0}{\gamma c^2}(n + n_{\text{st}})\right]\left(\frac{Z}{r_0}\right), \tag{37}$$

$$\frac{\Omega_{\theta 0} r_0}{c^2\gamma}\dot{Z} = \left[-\frac{2\nu}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{a}\right)^2 + \frac{M_r \Omega_{\theta 0}}{\gamma^2 e c^2} + \frac{u_\parallel^2}{c^2} - \frac{u_\parallel \Omega_{z0} r_0}{\gamma c^2}(-n_{\text{st}} + n)\right]\left(\frac{X}{r_0}\right) - \Delta, \tag{38}$$

where

$$n_{\text{st}} \approx \mu^2/bm,$$

$$b = B_{\theta 0}/B_{z0}, \quad m = -2\alpha r_0,$$

$$\mu = \Omega_s \epsilon_{\text{st}} \alpha r_0/\Omega_{z0}, \tag{39}$$

and

$$\Delta = \beta_\parallel^2 - r_0 \Omega_{z0} u_\parallel / c^2\gamma + \Omega_{\theta 0} M_r/\gamma^2 e c^2.$$

Equations (37) and (38) describe the linear transverse motion of the guiding center of the beam centroid in a MBA, for a short time interval after injection of the beam.

The guiding center of the gyrating centroid is on the minor axis when $\Delta = 0$. This condition provides a simple relation on the ratio of the vertical magnetic B_{z0} required to keep the guiding center on the minor axis when its centroid has transverse velocity to the vertical magnetic field B_{z0}^* required to keep the g.c. on the axis in the absence of transverse velocity. If $\kappa \equiv u_{\perp 0}/u_0$, where $u_0^2 = u_{\parallel 0}^2 + u_{\perp 0}^2$, the ratio of the two fields is

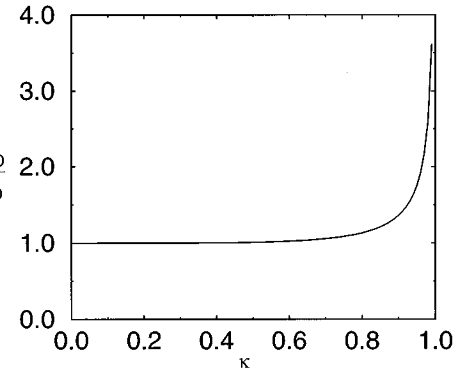

FIG. 2. Plotting of Eq. (40).

$$\frac{B_{z0}}{B_{z0}^*} = \frac{1 - \kappa^2/2}{(1 - \kappa^2)^{1/2}}. \tag{40}$$

Equation (40) is plotted in Fig. 2. For small values of κ, the ratio B_{z0}/B_{z0}^* is nearly unity but increases dramatically as $\kappa \to 1$. It is interesting and not intuitively obvious that the g.c. remains on its initial equilibrium position as $u_\perp$ increases provided that $u_\perp \lesssim u_0/2$.

As $u_\perp$ grows at the expense of u_θ, it would be expected that the guiding center would start moving inwards towards the major axis, since the force from the betatron field varies linearly with u_θ, while the centrifugal force has a quadratic dependence on u_θ. However, this is not the case because a new force is developed from the fast motion that compensates for this faster reduction of the centrifugal force.

Figure 3 shows a symmetric fast orbit with its guiding center on the minor axis. The B_z field provides a radial force F_b on the electron that is $F_b = -eu_\theta B_{z0}(1 - nx/r_0)$. When this force is averaged over the fast orbit gives, for $u_\perp^2/u_0^2 \ll 1$,

$$\langle F_b \rangle = -eB_{z0}u_0\left(1 - \frac{u_\perp^2}{2u_0^2}\right). \tag{41}$$

The radial component of the force that is associated with the fast motion is $F_f^{\text{rad}} = -eu_\perp B_{\theta 0}(r_0/r)\cos\phi$. When this force is averaged over the fast orbit becomes

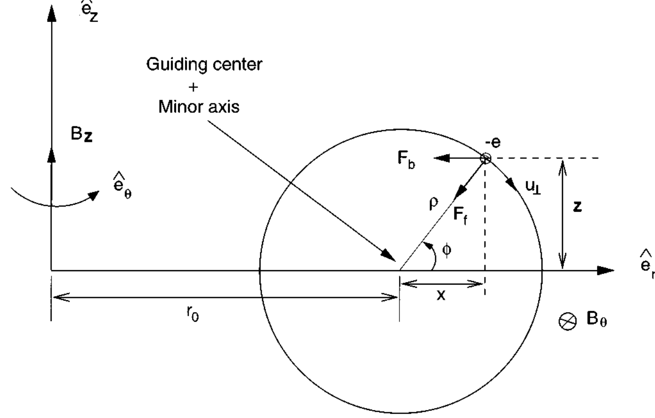

FIG. 3. A symmetric fast orbit with its g.c. on the minor axis.

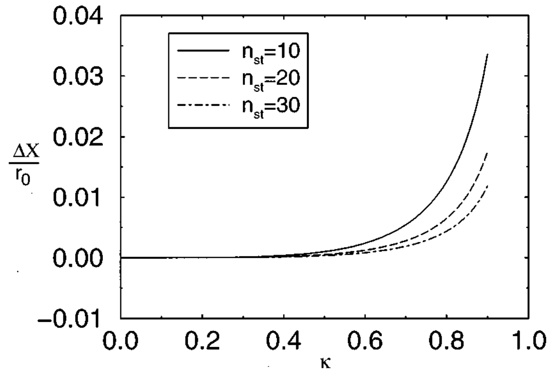

FIG. 4. $\Delta X/r_0$ vs κ for different values of n_{st}.

$$\langle F_f^{\mathrm{rad}}\rangle = m\Omega_{z0}\frac{u_\perp^2}{2u_0^2}. \tag{42}$$

Combining the centrifugal force $-\gamma m u_\theta^2/r_0$ with $-\langle F_f^{\mathrm{rad}}\rangle$, we get exactly $\langle F_b\rangle$ of Eq. (41).

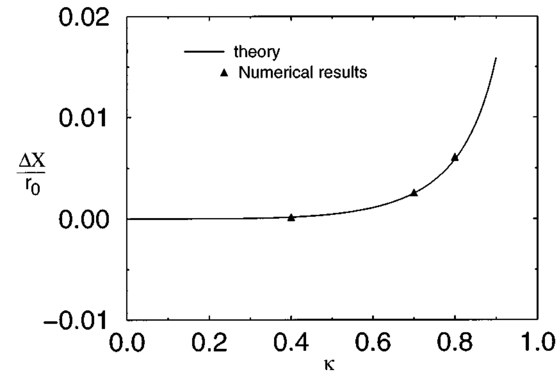

FIG. 5. Comparison of theory [Eq. (44)] with the numerical results for three values of $u_\perp/u_0$, when $B_{\theta0}=3$ kG, $B_{z0}=27.52$ G, $I_{\mathrm{st}}=25$ kA, $\gamma=2.2$, and $n_{\mathrm{st}}=22.17$.

If the vertical magnetic field is kept constant while κ increases, the guiding center will move radially away from the minor axis. The displacement can be determined from Eq. (38), which gives

$$\frac{\Delta X}{r_0} = \frac{\Delta}{[\beta_\parallel^2 + \Omega_{\theta0}M_r/\gamma^2 ec^2 - (2\nu/\gamma\gamma_\parallel^2)(r_0/a)^2 + (\beta_\parallel\Omega_{z0}r_0/\gamma c)(n_{\mathrm{st}}-n)]}. \tag{43}$$

If the value of the vertical magnetic field has been selected to make the minor axis equilibrium position for the reference electron on the beam centroid, injected with $u_\perp=0$, the equilibrium position will shift off the minor axis when the same electron will be injected with $u_\perp\neq 0$, but with the same total velocity u_0. Under these conditions and for $\nu=0$, Eq. (43) gives

$$\frac{\Delta X}{r_0} = \frac{(1-\kappa^2/2) - \sqrt{1-\kappa^2}}{(n_{\mathrm{st}}-n)\sqrt{1-\kappa^2} + (1-\kappa^2/2)}. \tag{44}$$

Figure 4 shows $\Delta X/r_0$ vs κ for $n=1/2$ and different values of n_{st}. Comparison of theory with numerical results is shown in Fig. 5. The agreement is very good.

Since γ is independent of time, the bounce frequency can be easily computed by taking the time derivative of Eq. (37) and using Eq. (38). The result is

$$\omega_b^2 = \left[-\frac{2\nu}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{\mathrm{a}}\right)^2 + \frac{u_\parallel\Omega_{z0}r_0}{\gamma c^2}(n+n_{\mathrm{st}})\right] \times\left[-\frac{2\nu}{\gamma\gamma_\parallel^2}\left(\frac{r_0}{a}\right)^2 + \frac{\Omega_{\theta0}M_r}{\gamma^2 ec^2} + \frac{u_\parallel^2}{c^2} + \frac{u_\parallel\Omega_{z0}r_0}{\gamma c^2}(n_{\mathrm{st}}-n)\right] \times\left(\frac{\Omega_{\theta0}r_0^2}{c^2\gamma}\right)^{-2}. \tag{45}$$

The fast velocity term $\Omega_{\theta0}M_r/\gamma^2 ec^2$ is equal to $u_\perp^2/2c^2$, which typically is substantially smaller than unity, while the largest term $u_\parallel\Omega_{z0}r_0/\gamma c^2 n_{\mathrm{st}}\approx n_{\mathrm{st}}$, which shortly after injection is greater that 10. Therefore the fast velocity has only a minor effect on the bounce frequency during the initial time period following beam injection.

IV. NUMERICAL RESULTS

An implicit assumption of the theory is that the relativistic invariance M_r is an approximate constant of the motion. To test the validity of this hypothesis, we carried out a series of computer runs in a MBA configuration. The instantaneous value of M_r is computed from

$$M_r = \frac{P_\perp^2}{2m_0 B},$$

where $P_\perp$ is the component of the electron momentum transverse to the total magnetic field and B is the magnitude of the total magnetic field that includes all three fields, i.e., toroidal, betatron, and strong focusing.

The momentum is computed from the solution of the relativistic equations of motion using a fourth order Runge-Kutta method with a set of initial conditions. The components of the total magnetic field are computed at the guiding center of the reference electron using the same technique as that described in Ref. [11]. The components of the induced electric field are computed from the time varying magnetic field. In all the runs, the reference electron was injected near the mi-

TABLE I. Results of computer test runs.

Run	Time (μsec)	$u_\perp/c$	$B_{\theta 0}$ (G)	$M_r \times 10^{15}$ $\frac{\text{kg m}^2/\text{sec}^2}{\text{T}}$
	0.0	1.983×10^{-2}	3000	2.54
A	9.9	1.993	3398	2.27
	19.9	2.041	3796	2.13
	0.0	3.315	6000	1.31
B	9.9	3.430	6398	1.31
	19.9	3.440	6796	1.25

nor axis with $u_\perp = 0$. To increase its transverse velocity during acceleration, the stellarator windings were randomly displaced from their ideal position by ± 6 mm, while B_θ was considered to be independent of time. When the desired $u_\perp$ was obtained, the B_θ field started to increase linearly with time, while $dB_z/dt = 0$ and the stellarator windings were brought back to their ideal position.

Results from two computer runs are shown in Table I for two different initial values of B_θ. The value of M_r shown in Table I is computed using the total magnetic field at the instantaneous guiding center at the specific times shown in the first column of the table. The transverse velocity is computed at the rotating frame of reference. $t=0$ is defined as the moment when $dB_z/dt = 0$. In run A, the toroidal field on the minor axis varies as $B_{\theta 0} = 3$ kG$+(40$ G/μsec$)t$ with $B_z = 114.14$ G$=$const and $\gamma = 6.87 =$const. In run B, $B_{\theta 0} = 6$ kG$+(40$ G/μsec$)t$ with $B_z = 229.46$ G$=$const and $\gamma = 13.23 =$const. The computer results indicate that M_r varies less than 16%. In essence, the adiabatic conditions [12] $(1/B_\theta \Omega_{\theta 0})|\partial B_\theta/\partial t| \ll 1$ and $(\rho/B)|\partial B/\partial r| \ll 1$ are satisfied by a large margin. Specifically, the second condition can be written as $\rho/L \ll 1$, where ρ is the Larmor radius and L is the scale length of the numerical model and the NRL experiment. Since for run A $\rho = 0.8$ mm and $L = 100$ mm, the ratio ρ/L is less than 1%. The margin is even larger for the temporal adiabatic condition.

Neither is it surprising that γ remains approximately constant while B_θ increases with time, because

$$\frac{d\gamma}{dt} = -\frac{e}{m_0 c^2}\vec{v}\cdot\vec{E} = -\frac{e}{m_0 c^2} v_\phi E_\phi = +\frac{e}{m_0 c^2} v_\phi \frac{d\Phi}{dt},$$

where v_ϕ is the poloidal velocity component and Φ is the magnetic flux through the transverse orbit of the reference electron. Since $d\Phi/dt \simeq 0$ for an adiabatic system [10], then $\gamma \approx$ const.

V. SUMMARY AND CONCLUSIONS

In this paper we have derived the equations that describe the slow motion (bounce) of the reference electron, which is located at the beam centroid, taking into account its fast motion, for an electron beam that is confined in the magnetic field of a Modified Betatron Accelerator.

The two equations for the guiding center (g.c.) of the centroid have been derived by averaging the centroid orbit over two time periods that are distinctly different. These two periods are associated with the normal modes of the system. Although our approach is not exact, it has the advantage that the fast velocity appears explicitly in the various expressions and thus its significance on the centroid dynamics can be rather easily assessed.

A general conclusion of our results is that the fast velocity does not have a profound impact on the dynamics of the centroid, provided that the ratio $u_\perp/u_0 \lesssim 0.5$, where $u_\perp$ is the transverse and u_0 is the total velocity. A very interesting conclusion of our work is that the centroid equilibrium position is insensitive to $u_\perp$, provided that $u_\perp \lesssim u_0/2$. Obviously, such a conclusion is counterintuitive because as $u_\perp$ grows at the expense of u_θ, it would be expected that the guiding center would start moving inwards towards the major axis, since the force from the betatron field varies linearly with u_θ, while the centrifugal force has a quadratic dependence on u_θ. However, this is not the case because, as shown in Sec. III, a new force is developed from the fast motion that compensates for this faster reduction of the centrifugal force.

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